Fourier Series

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Joseph Fourier

Fourier was obsessed with the physics of heat and developed the Fourier series and transform to model heat-flow problems.

Joseph Fourier 1768 - 1830

Introduction

A Fourier series is an expansion of a periodic function $f(t)$ in terms of an infinite sum of cosines and sines

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

In other words, any periodic function can be resolved as a summation of constant value and cosine and sine functions:

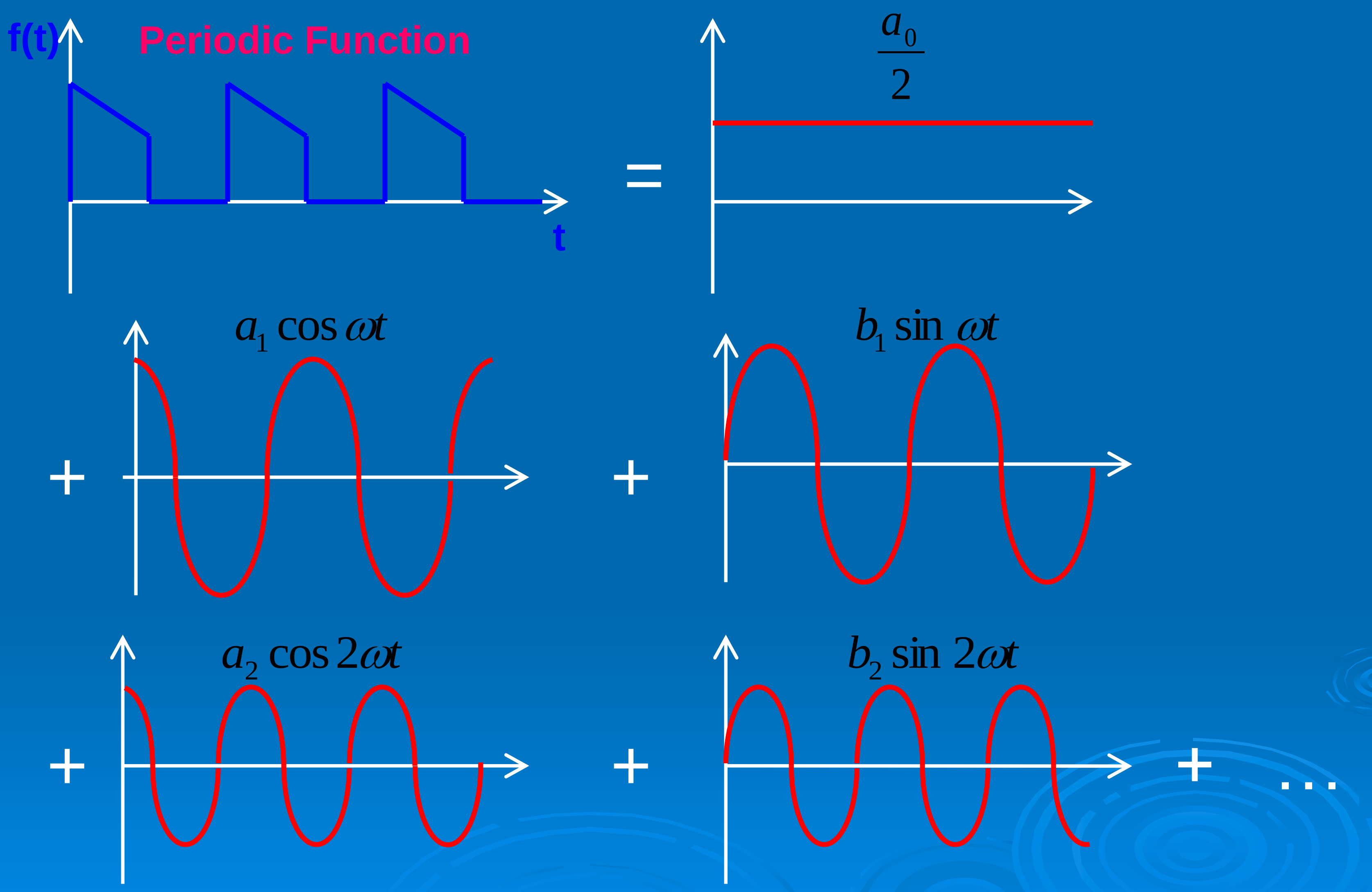
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{a_0}{2} \quad + (a_1 \cos \omega t + b_1 \sin \omega t)$$

$$+ (a_2 \cos 2\omega t + b_2 \sin 2\omega t)$$

$$+ (a_3 \cos 3\omega t + b_3 \sin 3\omega t) \quad + \ldots$$

The computation and study of Fourier series is known as *harmonic analysis* and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical.


f(t)
Periodic Function
t
=
$\frac{a_0}{2}$
+
$a_1 \cos \omega t$
+
$b_1 \sin \omega t$
+
$a_2 \cos 2\omega t$
+
$b_2 \sin 2\omega t$
+ ...

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

where $\omega = \frac{2\pi}{T} = \text{Fundementa l frequency}$

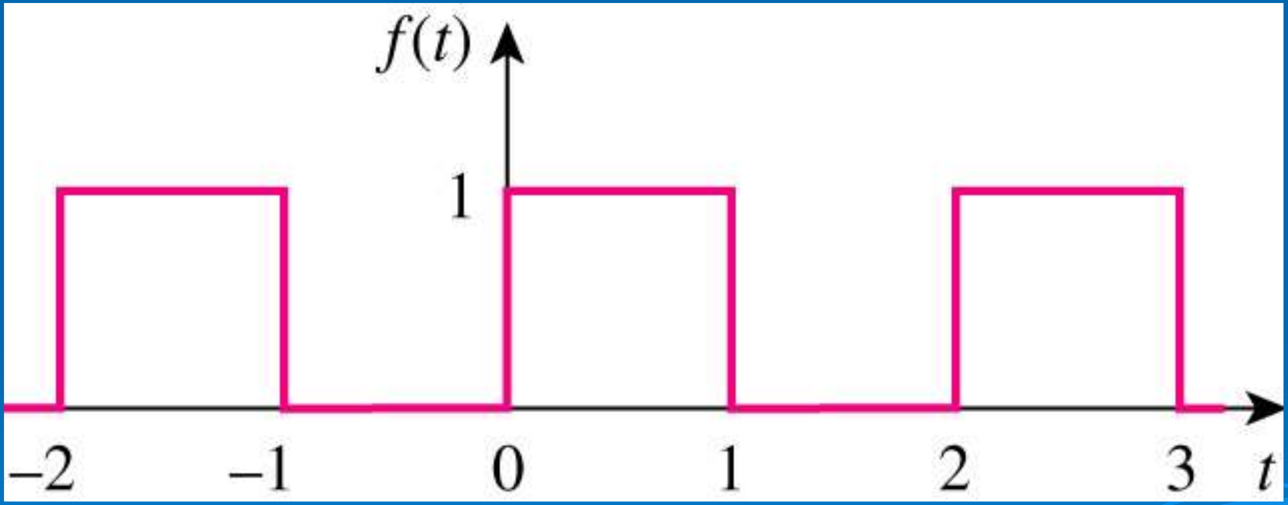
$$\boxed{a_0 = \frac{2}{T} \int_0^T f(t)dt}$$

$$\boxed{a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega t dt}$$

$$\boxed{b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega t dt}$$

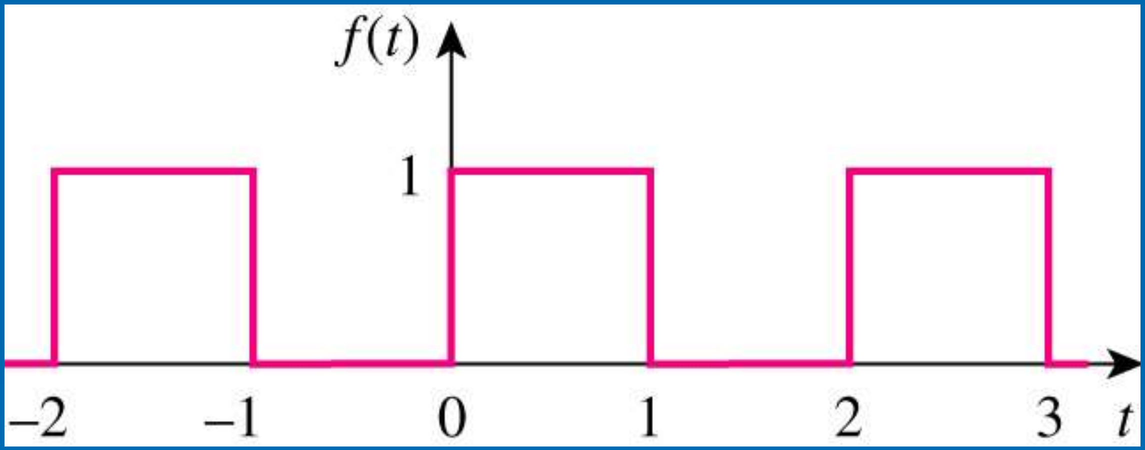
Example 1

Determine the Fourier series representation of the following waveform.



Solution

First, determine the period & describe the one period of the function:



$\longleftarrow T = 2 \longrightarrow$

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$$

Then, obtain the coefficients a_0, a_n and b_n:

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{2}\int_0^2 f(t)dt = \int_0^1 1dt + \int_1^2 0dt = 1 - 0 = 1$$

Or, since $\int_a^b f(t)dt$ is the total area below graph $y = f(t)$ over the interval $[a,b]$, hence

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{T}\times\begin{pmatrix}\text{Area below graph}\\ \text{over } [0,T]\end{pmatrix} = \frac{2}{2}\times(1\times1) = 1$$

$$a_n = \frac{2}{T}\int_0^2 f(t)\cos n\omega t dt$$

$$= \int_0^1 1\cos n\pi t dt + \int_1^2 0 dt = \left[\frac{\sin n\pi t}{n\pi}\right]_0^1 = \frac{\sin n\pi}{n\pi}$$

Notice that n is integer which leads $\sin n\pi = 0$, since $\sin \pi = \sin 2\pi = \sin 3\pi = \ldots = 0$

Therefore, $a_n = 0$.

$$b_n = \frac{2}{T}\int_0^2 f(t)\sin n\omega t\,dt$$

$$= \int_0^1 1\sin n\pi t\,dt + \int_1^2 0\,dt = \left[-\frac{\cos n\pi t}{n\pi}\right]_0^1 = \frac{1-\cos n\pi}{n\pi}$$

Notice that $\cos\pi = \cos 3\pi = \cos 5\pi = \ldots = -1$

$\cos 2\pi = \cos 4\pi = \cos 6\pi = \ldots = 1$

or $\cos n\pi = (-1)^n$

Therefore, $b_n = \frac{1-(-1)^n}{n\pi} = \begin{cases} 2/n\pi & , \quad n \text{ odd} \\ 0 & , \quad n \text{ even} \end{cases}$

Finally,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n\pi} \right] \sin n\pi t$$

$$= \frac{1}{2} + \frac{2}{\pi} \sin \pi t + \frac{2}{3\pi} \sin 3\pi t + \frac{2}{5\pi} \sin 5\pi t + \ldots$$

Comments

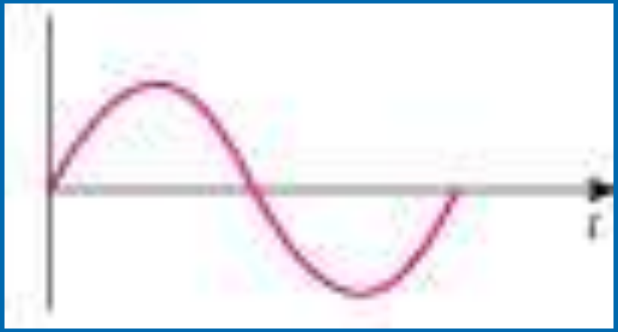
- The sum of the Fourier series terms can evolve (progress) into the original waveform

- From Example 1, we obtain

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \ldots$$

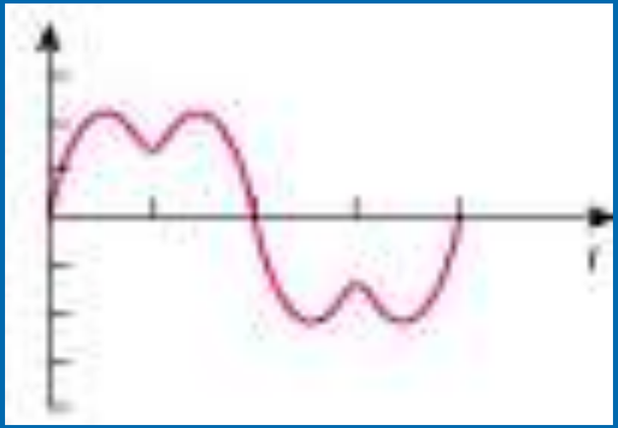
- It can be demonstrated that the sum will lead to the square wave:

(a)



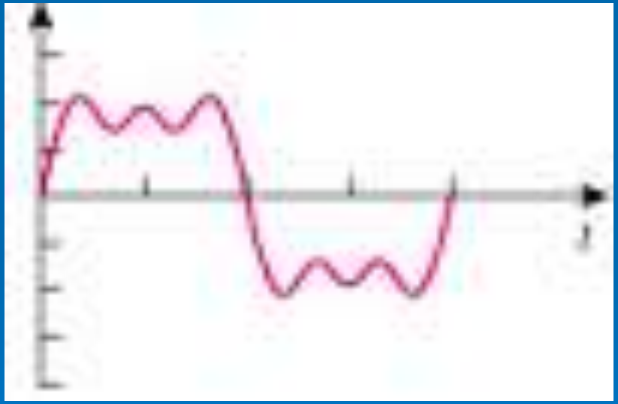
$$\frac{2}{\pi}\sin \pi t$$

(b)



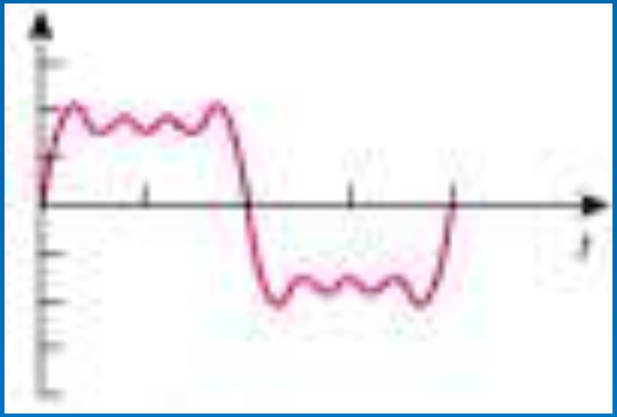
$$\frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t$$

(c)



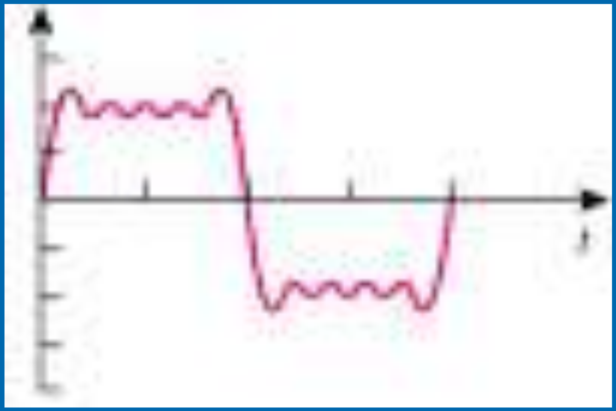
$$\frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t$$

(d)



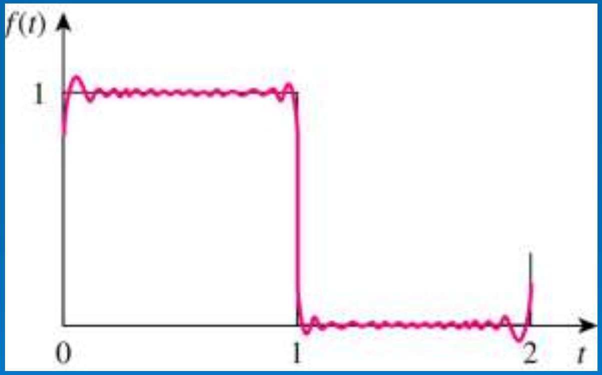
$$\frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \frac{2}{7\pi}\sin 7\pi t$$

(e)

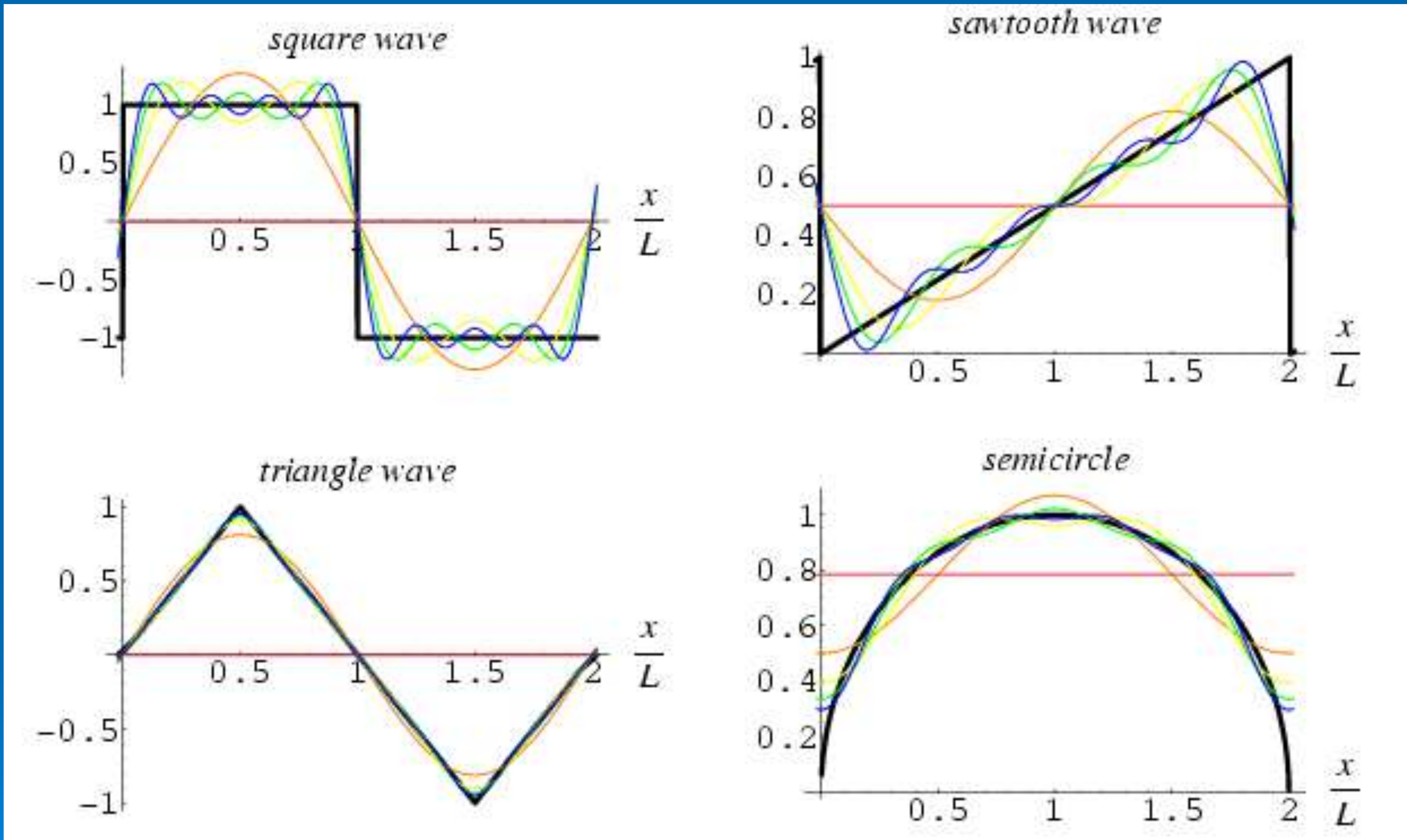


$$\frac{2}{\pi}\sin \pi t+\frac{2}{3\pi}\sin 3\pi t+\frac{2}{5\pi}\sin 5\pi t+\frac{2}{7\pi}\sin 7\pi t+\frac{2}{9\pi}\sin 9\pi t$$

(f)



$$\frac{1}{2}+\frac{2}{\pi}\sin \pi t+\frac{2}{3\pi}\sin 3\pi t+\ldots+\frac{2}{23\pi}\sin 23\pi t$$


square wave
sawtooth wave
triangle wave
semicircle
x/L
0.5
1
1.5
2
-0.5
-1
0.2
0.4
0.6
0.8

Example 2

Given $f(t) = t, \quad -1 \le t \le 1$

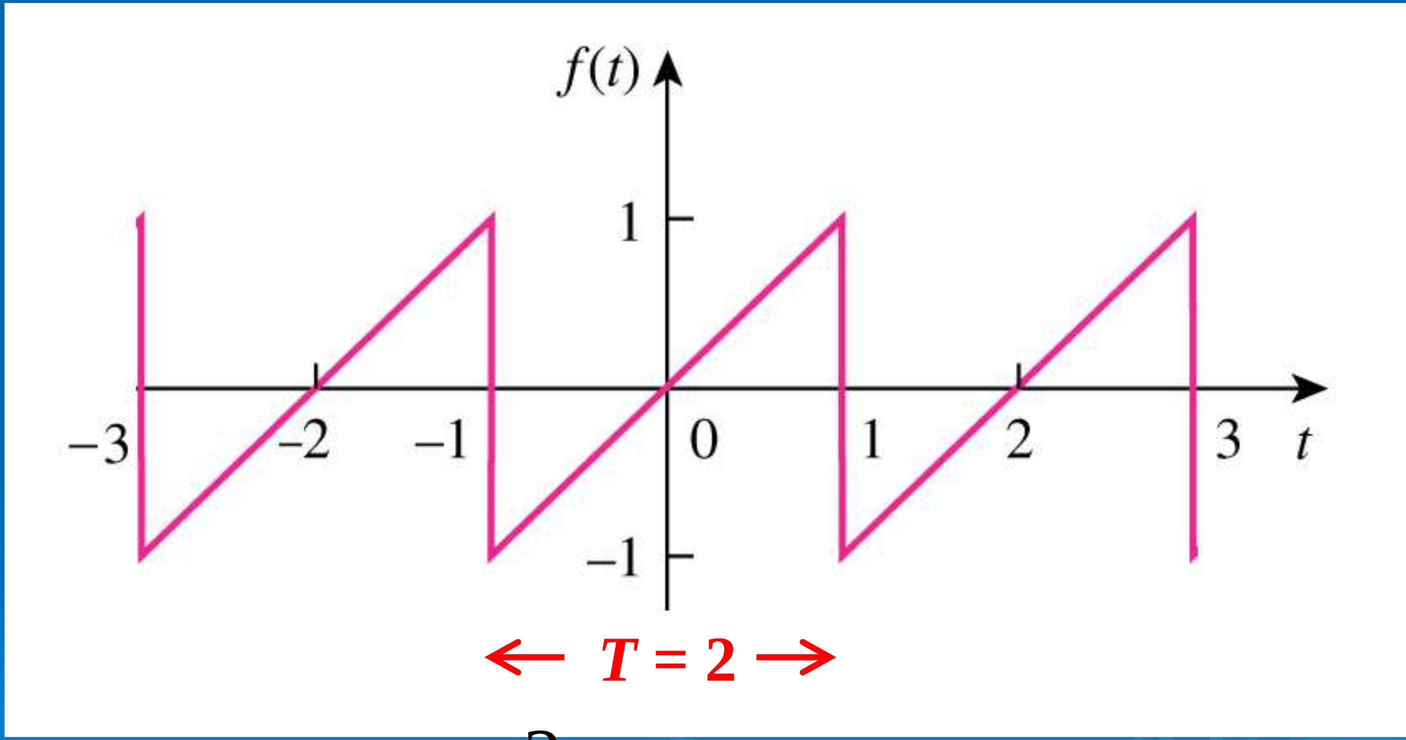
$$f(t+2) = f(t)$$

Sketch the graph of $f(t)$ such that $-3 \le t \le 3$.

Then compute the Fourier series expansion of $f(t)$.

Solution

The function is described by the following graph:



We find that $\omega = \frac{2\pi}{T} = \pi$

Then we compute the coefficients:

$$a_0 = \frac{2}{T}\int_{-1}^{1} f(t)dt$$

$$= \frac{2}{2}\int_{-1}^{1} tdt = \left[\frac{t^2}{2}\right]_{-1}^{1} = \frac{1-1}{2} = 0$$

$$a_n = \frac{2}{T}\int_{-1}^{1} f(t)\cos n\omega t\,dt = \int_{-1}^{1} t\cos n\pi t\,dt$$

$$= \left[\frac{t\sin n\pi t}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1}\frac{\sin n\pi t}{n\pi}dt$$

$$= \frac{\sin n\pi - [-\sin(-n\pi)]}{n\pi} + \left[\frac{\cos n\pi t}{n^2\pi^2}\right]_{-1}^{1}$$

$$= 0 + \frac{\cos n\pi - \cos(-n\pi)}{n^2\pi^2}$$

$$= \frac{\cos n\pi - \cos n\pi}{n^2\pi^2} = 0$$ since $\cos(-x) = \cos x$

$$b_n = \frac{2}{T}\int_{-1}^{1} f(t)\sin n\omega t dt = \int_{-1}^{1} t\sin n\pi t dt$$

$$= \left[-\frac{t\cos n\pi t}{n\pi}\right]_{-1}^{1} + \int_{-1}^{1}\frac{\cos n\pi t}{n\pi} dt$$

$$= \frac{-\cos n\pi + [-\cos(-n\pi)]}{n\pi} + \left[\frac{\sin n\pi t}{n^2\pi^2}\right]_{-1}^{1}$$

$$= -\frac{2\cos n\pi}{n\pi} + \frac{\sin n\pi - \sin(-n\pi)}{n^2\pi^2}$$

$$= -\frac{2\cos n\pi}{n\pi} = -\frac{2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}$$

Finally,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin n\pi t$$

$$= \frac{2}{\pi} \sin \pi t - \frac{2}{2\pi} \sin 2\pi t + \frac{2}{3\pi} \sin 3\pi t - \ldots$$

Symmetry Considerations

- Symmetry functions:

  (i) **even** symmetry

  (ii) **odd** symmetry

Even Symmetry

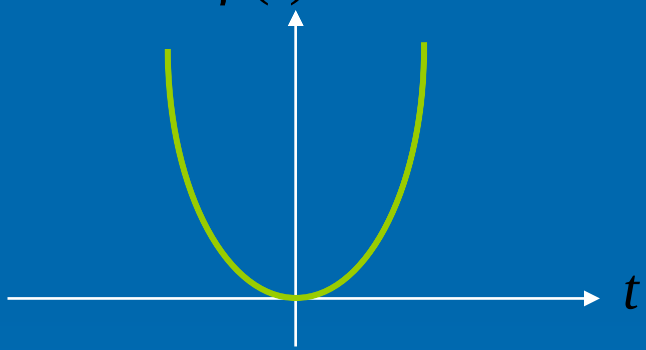
- Any function $f(t)$ is **even** if its plot is symmetrical about the vertical axis, i.e.

$$f(-t) = f(t)$$

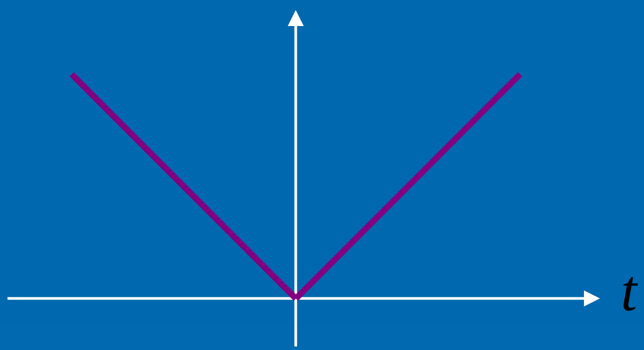
Even Symmetry

- The examples of **even** functions are:

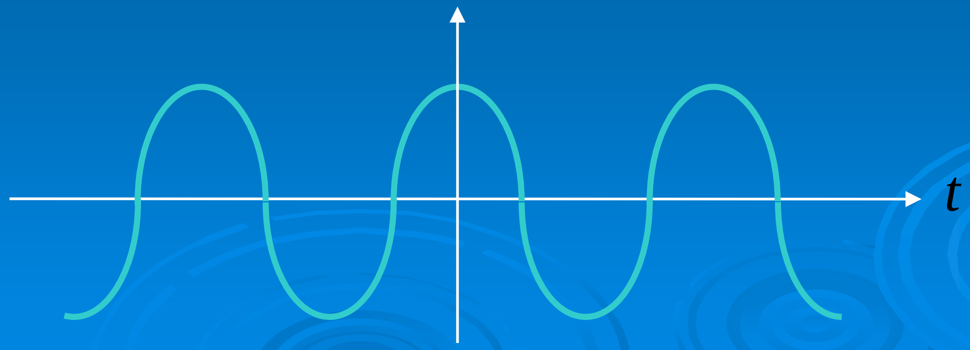
$f(t) = t^2$



$f(t) = |t|$

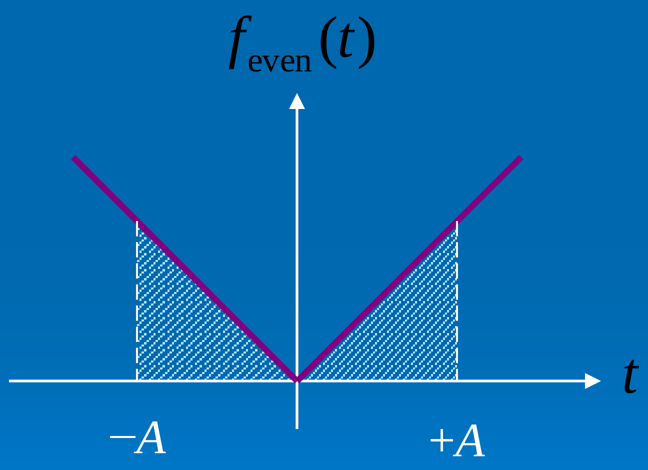


$f(t) = \cos t$



Even Symmetry

- The integral of an **even** function from $-A$ to $+A$ is twice the integral from 0 to $+A$



$$\int_{-A}^{+A} f_{\text{even}}(t)\,dt = 2\int_{0}^{+A} f_{\text{even}}(t)\,dt$$

Odd Symmetry

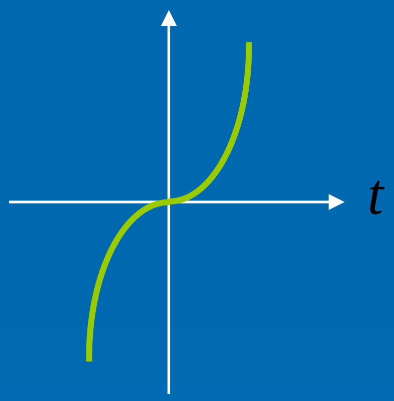
- Any function $f(t)$ is **odd** if its plot is antisymmetrical about the vertical axis, i.e.

$$f(-t) = -f(t)$$

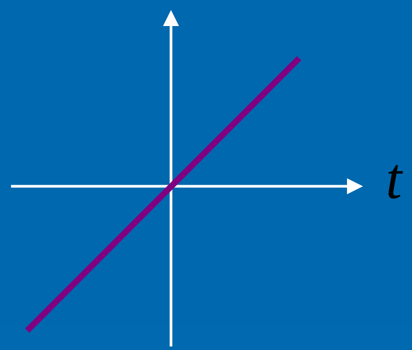
Odd Symmetry

- The examples of **odd** functions are:

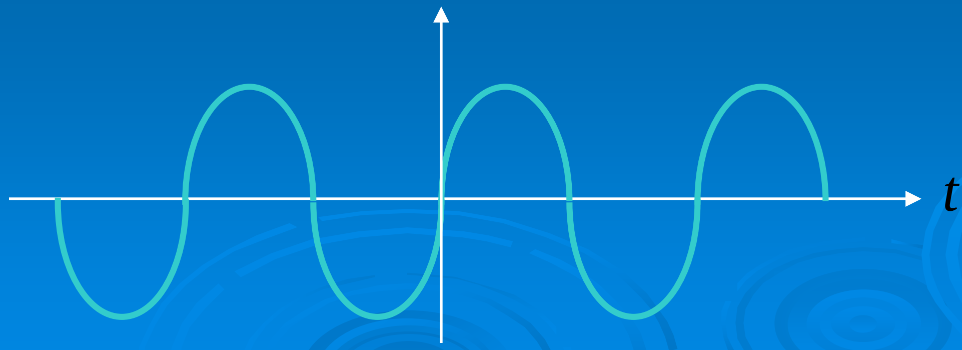
$$f(t) = t^3$$



$$f(t) = t$$

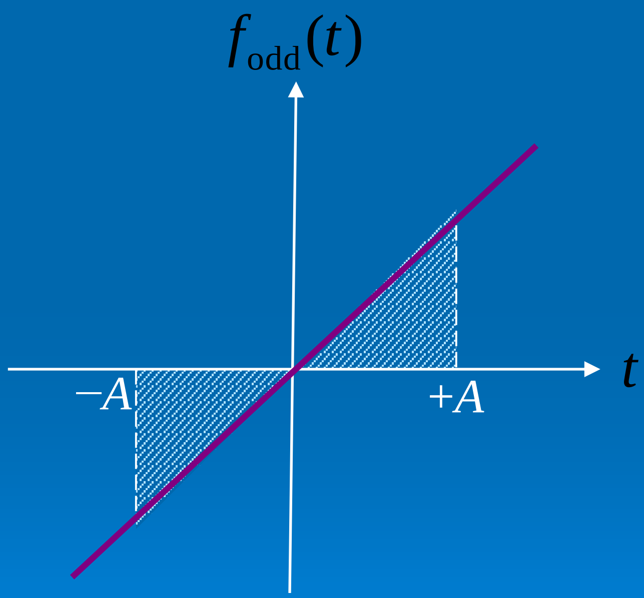


$$f(t) = \sin t$$



Odd Symmetry

- The integral of an **odd** function from $-A$ to $+A$ is zero



$$\int_{-A}^{+A} f_{\text{odd}}(t)\,dt = 0$$

Symmetry consideration

From the properties of even and odd functions, we can show that:

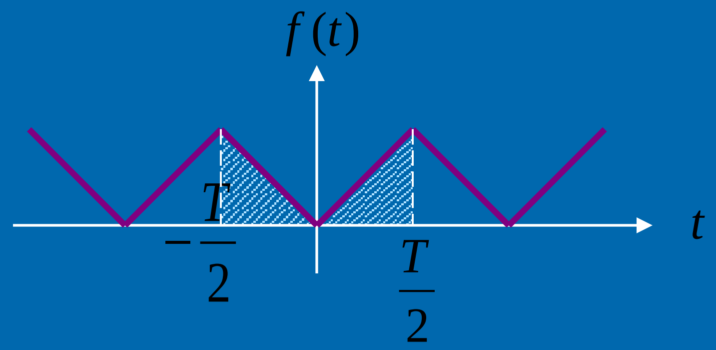
- for even periodic function;

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt \qquad b_n = 0$$

- for odd periodic function;

$$a_0 = a_n = 0 \qquad b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\,dt$$

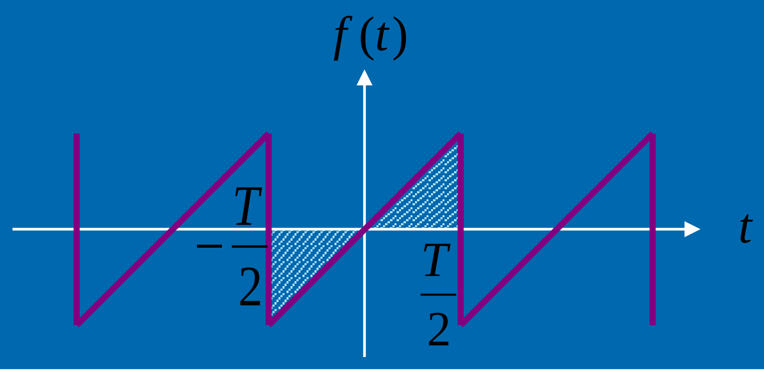
Even Function



$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f(t)}_{\text{(even)}} \times \underbrace{\cos n\omega t}_{\text{(even)}}}_{=\ \text{(even)}}\, dt = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega t\, dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f(t)}_{\text{(even)}} \times \underbrace{\sin n\omega t}_{\text{(odd)}}}_{=\ \text{(odd)}}\, dt = 0$$

Odd Function



$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f(t)}_{\text{(odd)}}\,dt = 0$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f(t)}_{\text{(odd)}} \times \underbrace{\cos n\omega t}_{\text{(even)}}}_{=\ \text{(odd)}}\,dt = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f(t)}_{\text{(odd)}} \times \underbrace{\sin n\omega t}_{\text{(odd)}}}_{=\ \text{(even)}}\,dt = \frac{4}{T}\int_{0}^{T/2} f(t)\sin n\omega t\,dt$$

Example 3

Given $$f(t)=\begin{cases}-1 & , \quad -2<t<-1 \\ t & , \quad -1<t<1 \\ 1 & , \quad 1<t<2\end{cases}$$

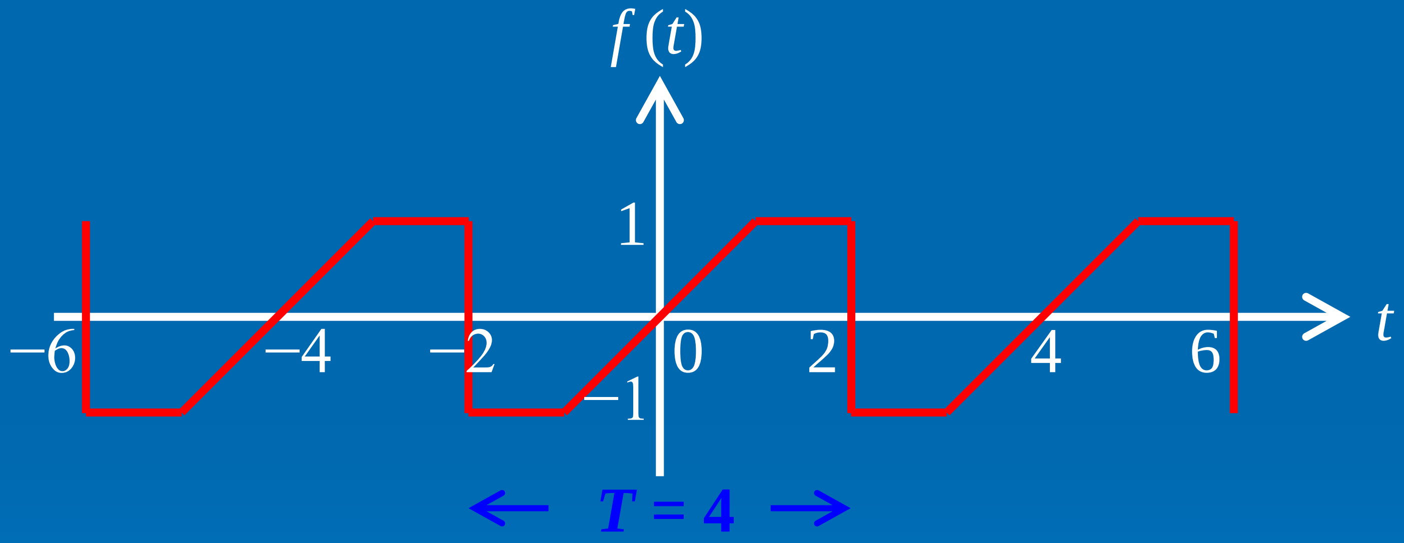
$$f(t+4)=f(t)$$

Sketch the graph of $f(t)$ such that $-6 \le t \le 6$.

Then compute the Fourier series expansion of $f(t)$.

Solution

The function is described by the following graph:



We find that $\omega = \frac{2\pi}{T} = \frac{\pi}{2}$

Then we compute the coefficients. Since $f(t)$ is an odd function, then

$$a_0 = \frac{2}{T}\int_{-2}^{2} f(t)dt = 0$$

and

$$a_n = \frac{2}{T}\int_{-2}^{2} f(t)\cos n\omega t dt = 0$$

$$b_n = \frac{2}{T}\int_{-2}^{2} f(t)\sin n\omega t dt = \frac{4}{T}\int_{0}^{2} f(t)\sin n\omega t dt$$

$$= \frac{4}{4}\left[\int_{0}^{1} t\sin n\omega t dt + \int_{1}^{2} 1\sin n\omega t dt\right]$$

$$= \left[-\frac{t\cos n\omega t}{n\omega}\right]_0^1 + \int_0^1 \frac{\cos n\omega t}{n\omega}dt + \left[-\frac{\cos n\omega t}{n\omega}\right]_1^2$$

$$= -\frac{\cos n\omega}{n\omega} + \left[\frac{\sin n\omega t}{n^2\omega^2}\right]_0^1 - \frac{\cos 2n\omega - \cos n\omega}{n\omega}$$

$$= -\frac{\cos 2n\omega}{n\omega} + \frac{\sin n\omega}{n^2\omega^2} = -\frac{2\cos n\pi}{n\pi}$$

since $\sin 2n\omega = \sin n\pi = 0$

Finally,

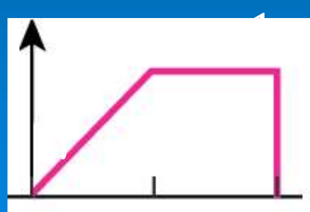
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

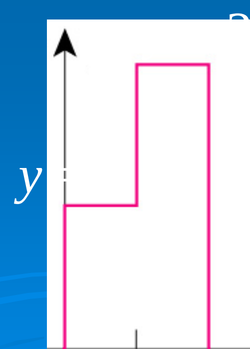
$$= \sum_{n=1}^{\infty} \left(-\frac{2\cos n\pi}{n\pi} \right) \sin \frac{n\pi t}{2}$$

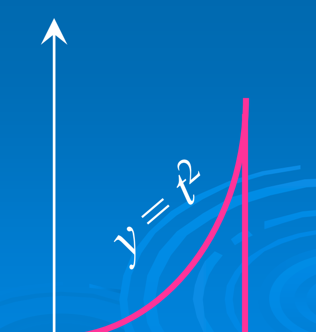
$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin \frac{n\pi t}{2}$$

Function defined over a finite interval

- Fourier series only support periodic functions
- In real application, many functions are non-periodic
- The non-periodic functions are often can be defined over finite intervals, e.g.

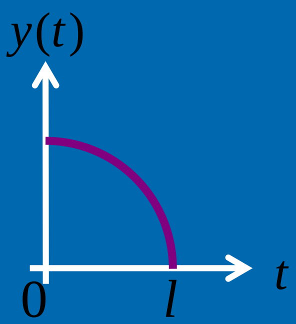






- Therefore, any non-periodic function must be **extended to a periodic function** first, before computing its Fourier series representation
- Normally, we prefer symmetry (even or odd) periodic extension instead of normal periodic extension, since symmetry function will provide zero coefficient of either a_n or b_n
- This can provide a simpler Fourier series expansion

Non-periodic function

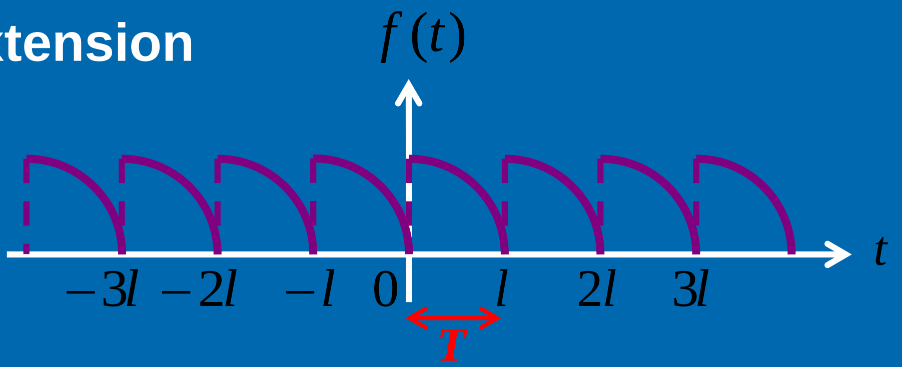


Periodic extension

$$f(t) = y(t) \quad , \quad 0 < t < l$$

$$f(t+l) = f(t)$$

$$T = l$$

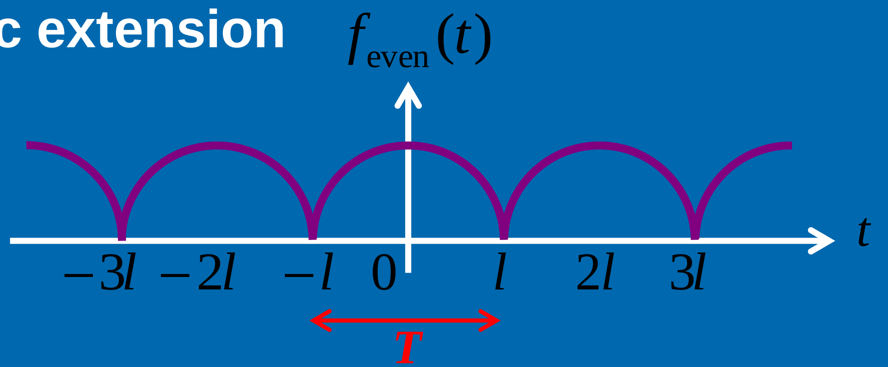


Even periodic extension

$$f(t) = \begin{cases} y(t) & , \quad 0 < t < l \\ y(-t) & , \quad -l < t < 0 \end{cases}$$

$$f(t+2l) = f(t)$$

$$T = 2l$$

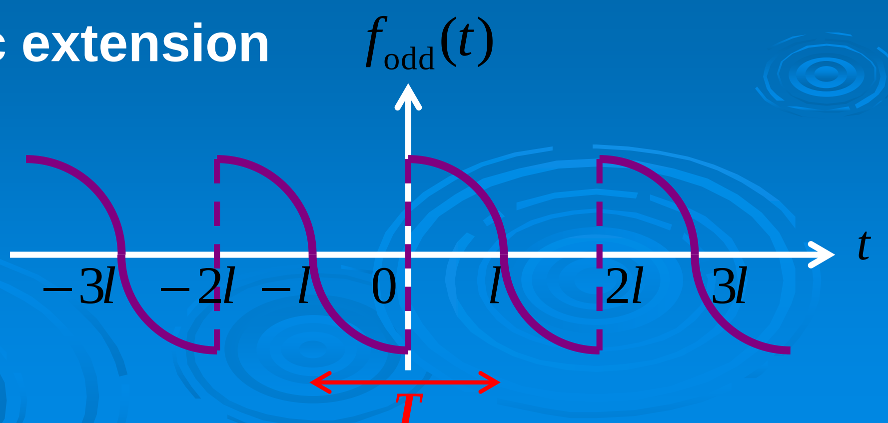


Odd periodic extension

$$f(t) = \begin{cases} y(t) & , \quad 0 < t < l \\ -y(-t) & , \quad -l < t < 0 \end{cases}$$

$$f(t+2l) = f(t)$$

$$T = 2l$$



Half-range Fourier Series Expansion

- The Fourier series of the even or odd periodic extension of a non-periodic function is called as the *half-range Fourier series*
- This is due to the non-periodic function is considered as the half-range before it is extended as an even or an odd function

- If the function is extended as an even function, then the coefficient $b_n = 0$, hence

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

which only contains the cosine harmonics.

- Therefore, this approach is called as the *half-range Fourier cosine series*

- If the function is extended as an odd function, then the coefficient $a_n = 0$, hence

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

which only contains the sine harmonics.

- Therefore, this approach is called as the *half-range Fourier sine series*

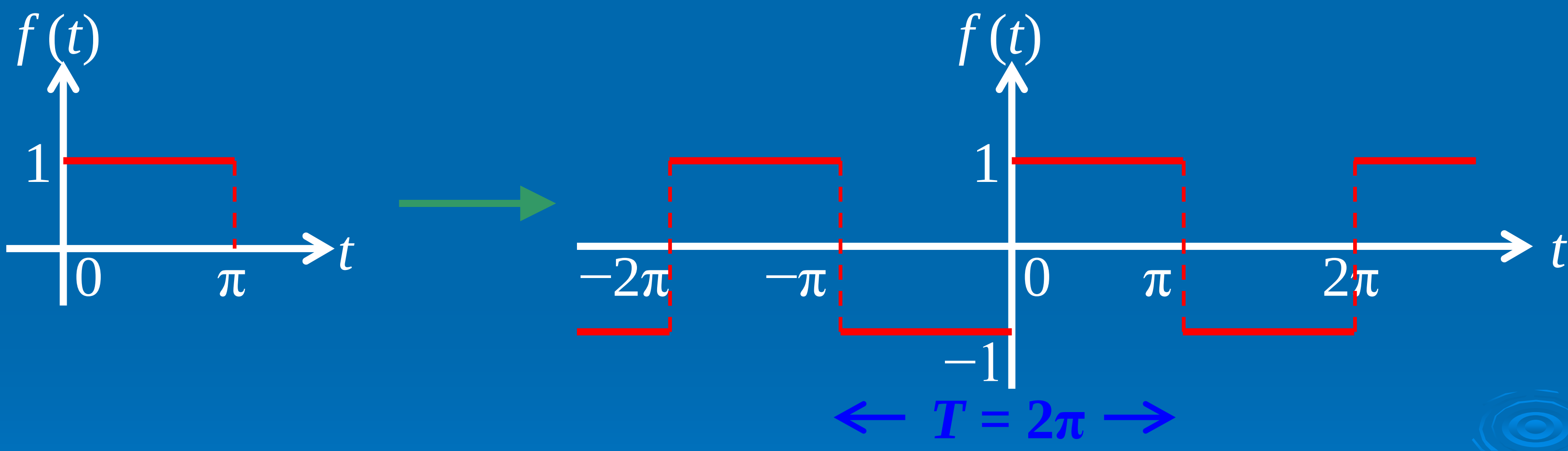
Example 4

Compute the half-range Fourier sine series expansion of $f(t)$, where

$$f(t) = 1 \quad , \quad 0 < t < \pi$$

Solution

Since we want to seek the half-range sine series, the function to is extended to be an odd function:



$$\omega = \frac{2\pi}{T} = 1$$

Hence, the coefficients are

$$a_0 = a_n = 0$$

and

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t dt = \frac{4}{2\pi}\int_0^{\pi} 1\sin nt dt$$

$$= \frac{2}{\pi}\left[-\frac{\cos nt}{n}\right]_0^{\pi} = \frac{2}{n\pi}(1-\cos n\pi) = \begin{cases} 4/n\pi & , \quad n \text{ odd} \\ 0 & , \quad n \text{ even} \end{cases}$$

Therefore,

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-\cos n\pi)\sin nt = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi}\sin nt$$

Thanks